The Optional Stopping/Sampling Theorem.

We'll confine ourselves to showing that in a fair game, a gambler cannot make money. The description of why it is not possible to beat an unfair game develops closely along the same lines. Before proving the OST, we'll have to define a few terms and explain how we can compute a betting strategy payoff.

Definition: By a discrete time martingale, we mean a stochastic process, $\{X_n\}_{n\geq 0}$ such that: (i) $E(X_n) < \infty$, for all $n \geq 0$ and (ii) $E(X_{n+1}|X_0 = a_0, \ldots, X_n = a_n) = a_n$, for all $n \geq 0$. Equivalently, we say that $\{X_n\}_{n\geq 0}$ is a martingale if $E(X_{n+1} - X_n|X_0 = a_0, \ldots, X_n = a_n) = 0$.

Definition: A sequence of random variables $\{H_n\}_{n\geq 1}$ is called a "betting sequence" with respect to a sequence $\{X_n\}_{n\geq 0}$ if for all $n \geq 1$, the value of H_n is decidable by $X_0, X_1, \ldots, X_{n-1}$ i.e. H_n is a function of $X_0, X_1, \ldots, X_{n-1}$ for all $n \geq 1$.

Consider a game where we flip a coin: assume that before each flip we bet a certain amount of money and if the flip comes up heads, we earn the amount we bet and if the flip comes up tails, we lose the amount we bet. More specifically: Let X_m denote the value (heads or tails) of the mth flip. Let H_m be the amount we bet on the mth flip. Let $\xi_m = 1$ if the mth flip is heads and $\xi_m = -1$ if the mth flip is tails. Then our total winnings up to and including flip n is given by:

$$W_n = W_0 + \sum_{m=1}^{n} H_m \xi_m$$

Let $Y_n = Y_0 + c_1\xi_1 + c_2\xi_2 + \ldots + c_n\xi_n$, be the total winnings of a gambler who bets exactly c_i dollars on the ith flip (with $Y_0 = W_0$), we have:

$$W_n = W_0 + \sum_{m=1}^{n} H_m c_m \xi_m = W_0 + \sum_{m=1}^{n} H_m[(Y_0 - Y_0) + (c_1\xi_1 - c_1\xi_1) + \ldots$$

$$\ldots + (c_{m-1}\xi_{m-1} - c_{m-1}\xi_{m-1}) + c_m\xi_m] = W_0 + \sum_{m=1}^{n} H_m(Y_m - Y_{m-1})$$

Using this sequence, we'll show that if our earnings sequence,$\{Y_n\}$, is a martingale with respect to our outcome sequence, $\{X_n\}$, then our winnings sequence, $\{W_n\}$, is also a martingale with respect to our outcome sequence, $\{X_n\}$.

Theorem: Suppose $\{Y_n\}$ is a martingale with respect to $\{X_n\}$ and that $\{H_n\}$ is a betting sequence with respect to $\{X_n\}$ with $0 \leq H_n \leq c \in R$ for all n, then $\{W_n\}$ is a martingale with respect to $\{X_n\}$.

Proof. Our earnings at time $n+1$ is given by: $W_{n+1} = W_n + H_{n+1}(Y_{n+1} - Y_n)$, which is $W_{n+1} - W_n = H_{n+1}(Y_{n+1} - Y_n)$. Then:

$$E(W_{n+1} - W_n | X_o, \ldots, X_n) = E(H_{n+1}(Y_{n+1} - Y_n) | X_o, \ldots, X_n)$$

Because $\{H_n\}$ is a betting sequence, the value of H_{n+1} is deterministic (given $X_o, \ldots, X_n$) thus we can treat its expectation like a constant and:

$$E(H_{n+1}(Y_{n+1} - Y_n) | X_o, \ldots, X_n) = H_{n+1} E(Y_{n+1} - Y_n) | X_o, \ldots, X_n)$$

Therefore the sequence $\{W_n\}$ is a martingale since it is equal to $\{Y_n\}$. □

Definition: We say that T is a "stopping time" with respect to a sequence $\{X_n\}$ if the occurrence of the event {we stop at time n} is decidable on $X_0, X_1, \ldots X_n$.

For example, say we decide to bet \$1 until time T at which point we stop betting. Then $H_m = 1$ for $m < T$ and $H_m = 0$ for $m \geq T$. Then: $\{H_m = 0\} = \{T > m\}^c = \{T \leq m\} = \cup_{k=1}^{m-1} \{T = k\}$, which is decidable on $X_0, X_1, \ldots, X_{m-1}$ since $\{T = k\}$ is decidable on $X_0, X_1, \ldots, X_k$. Then T is a stopping time. If this all sounds a bit too technical, it may be easier to think of a stopping time as a time which you can fully determine without looking into the future.

We'll now prove the optional stopping/sampling theorem. The OST essentially says that if our expected earnings at time n is zero given the first $n-1$ outcomes of the game, and our betting strategy at time n is completely determined by the first $n-1$ outcomes of the game. Then for any stopping time (i.e. any stopping decision decidable by the first n outcomes of the game), our expected winnings at time T is also equal to zero.

Incidently, the 'super/sub' martingale versions of OST say that if our expected earnings at time n is decreasing/increasing given the first $n-1$ outcomes of the game, and our betting strategy at time n is completely determined by the first $n-1$ outcomes of the game. Then for any stopping time, our expected winnings at time T is also decreasing/increasing.

Theorem: If $\{Y_n\}$ is a martingale with respect to $\{X_n\}$, T a stopping time with respect to $\{X_n\}$, and $\{H_n\}$ a betting sequence with respect to $\{X_n\}$, then the stopped process $\{Y_{\min\{T,n\}}\}$ is a martingale with respect to $\{X_n\}$, where $Y_{\min\{T,n\}} = Y_T$ if $T < n$ and $Y_{\min\{T,n\}} = Y_n$ otherwise.

Proof. To begin, we claim that $W_n = Y_0 + \sum_{m=1}^{n} H_m(Y_m - Y_{m-1}) = Y_{\min\{T,n\}}$. Consider two cases:

case (i): Assume $T > n$, then $H_m = 1, \forall m \leq n \implies$

$$W_n = Y_0 + \sum_{m=1}^{n} H_m(Y_m - Y_{m-1}) = Y_0 + \sum_{m=1}^{n} (Y_m - Y_{m-1}) =$$

$$= (Y_0 - Y_0) + (Y_1 - Y_1) + \ldots + (Y_{n-1} - Y_{n-1}) + Y_n = Y_n = Y_{\min\{T,n\}}$$

case (ii): Assume $T \geq n$, then $H_m = 1$, $\forall m \leq T$ and $H_m = 0$, $\forall m > T \implies$

$$W_n = Y_0 + \sum_{m=1}^{n} H_m(Y_m - Y_{m-1}) = Y_0 + \sum_{m=1}^{T} H_m(Y_m - Y_{m-1}) + \sum_{m=T+1}^{n} 0(Y_m - Y_{m-1}) =$$

$$= Y_0 + \sum_{m=1}^{T}(Y_m - Y_{m-1}) = (Y_0 - Y_0) + (Y_1 - Y_1) + \ldots + (Y_{T-1} - Y_{T-1}) + Y_T = Y_T = Y_{\min\{T,n\}}$$

Then, $W_n = Y_{\min\{T,n\}}$ and by our previous theorem we saw that $\{W_n\}$ was a martingale. Thus we see that the stopped process $\{Y_{\min\{T,n\}}\}$ is also a martingale.

$\square$

awesome.