Continuity In A Metric Space

Let $f : (X, d) \to (S, \rho)$ be a function and $a \in X$, $\alpha = f(a)$. Then the following are equivalent:

i. f is continuous at a.
ii. $\forall \epsilon > 0$, $f^{-1}(B(\alpha, \epsilon))$ contains a ball with center at a.
iii. $\alpha = \lim f(x_n)$ whenever $a = \lim x_n$.

Proof. **iii** $\to$ **i**. By contrapositive. Assume that f is not continuous at a. That is, assume that $\lim_{x \to a} f(x) \neq f(a)$. Then, there exists $\epsilon > 0$ such that for all $\delta > 0$ there is an $x \in X$ such that $d(x, a) < \delta$ and $\rho(f(x), f(a)) \geq \epsilon$. Then, for all $n \in \mathbb{N}$ there exists x_m such that $d(x_m, a) < 1/n$ and $\rho(f(x_m), f(a)) \geq \epsilon$. Then $x_n \to a \in X$ does not imply $f(x_n) \to f(a) \in S$, from which it follows that, if $\alpha = \lim f(x_n)$ whenever $a = \lim x_n$, then f is continuous at a.

i $\to$ **ii**. If f is continuous at a, then $\lim_{x \to a} f(x) = f(a)$ and for given $\epsilon > 0$ and $B(\alpha, \epsilon) \subset S$, there exists $\delta > 0$ and $B(a, \delta) \subset X$ such that whenever $x \in B(a, \delta)$, we have $f(x) \in B(\alpha, \epsilon)$. Then because this is true for all $x \in B(a, \delta)$, we have $x \in B(a, \delta) \implies f(x) \in B(\alpha, \epsilon)$, from which it follows that $f(B(a, \delta)) \subset B(\alpha, \epsilon)$. Then for any choice of $\epsilon > 0$, there exists $\delta > 0$ such that $B(a, \delta) \subset f^{-1}(B(\alpha, \epsilon))$.

ii $\to$ **iii**. Let $\epsilon_1 > 0$ be given, then there exists $\delta_1 > 0$ such that $B(a, \delta_1) \subset f^{-1}(B(\alpha, \epsilon_1))$, by assumption. Reapplying this assumption: for given ϵ_2 with $0 < \epsilon_2 < \epsilon_1$ there exists δ_2 with $0 < \delta_2 < \delta_1$ such that: $B(a, \delta_2) \subset f^{-1}(B(\alpha, \epsilon_2))$. Continuing on, for given ϵ_n with $0 < \epsilon_n < \epsilon_{n-1} < \ldots < \epsilon_1$ there exists δ_n with $0 < \delta_n < \delta_{n-1} < \ldots < \delta_1$ such that: $B(a, \delta_n) \subset f^{-1}(B(\alpha, \epsilon_n))$. Then, we have:

$$\begin{gathered} B(a, \delta_1) \subset f^{-1}(B(\alpha, \epsilon_1)) \\ B(a, \delta_2) \subset f^{-1}(B(\alpha, \epsilon_2)) \\ \vdots \\ B(a, \delta_n) \subset f^{-1}(B(\alpha, \epsilon_n)) \end{gathered}$$

with: $B(a, \delta_n) \subset \ldots \subset B(a, \delta_2) \subset B(a, \delta_1)$ and $B(\alpha, \epsilon_n) \subset \ldots \subset B(\alpha, \epsilon_2) \subset B(\alpha, \epsilon_1)$. Then for each ball about a and α, we have:

$$\begin{gathered} x_1 \in B(a, \delta_1) \implies f(x_1) \in (B(\alpha, \epsilon_1)) \\ x_2 \in B(a, \delta_2) \implies f(x_2) \in (B(\alpha, \epsilon_2)) \\ \vdots \\ x_n \in B(a, \delta_n) \implies f(x_n) \in (B(\alpha, \epsilon_n)) \\ \vdots \end{gathered}$$

and, clearly, for $x_n \to a \in X$, we see that $f(x_n) \to \alpha \in S$. $\square$