Theorem: Let $p(z)$ be a polynomial of degree n and let $R > 0$ be sufficiently large so that p does not vanish in $\{z : |z| \geq R\}$. Then, if $\gamma(t) = Re^{it}$, $0 \leq t \leq 2\pi$, it follows that:

$$\frac{1}{2\pi i}\int_\gamma \frac{p'(z)}{p(z)}dz = n$$

Proof. To begin, clearly p splits over $\mathbb{C}$, so we can write $p(z) = c\prod_{i=1}^n (z - a_i)$ where c is some constant and $a_i \in \mathbb{C}$ are the zeroes of the polynomial p. Next, notice that $\gamma(t) = Re^{it}$, $\leq t \leq 2\pi$ is a closed, rectifiable curve and that when $R > 0$ is large enough such that p does not vanish in $\{z : |z| \geq R\}$ the zeroes, $a_1, a_2, \dots a_n$ are not points on the path γ and further, the winding number, $n(\gamma, a_i) = 1$ for each $i \in [1, n]$. Then, differentiating p, we get:

$$p'(z) = \frac{d}{dz}[c\prod_{i=1}^{n}(z-a_i)] = c\frac{d}{dz}[(z-a_1)(z-a_2)\dots(z-a_n)] =$$

$$= c[(z-a_1)'(z-a_2)\dots(z-a_n)+(z-a_1)(z-a_2)'\dots(z-a_n)+\dots+(z-a_1)(z-a_2)\dots(z-a_n)'] =$$

$$= c[1(z-a_2)\dots(z-a_n)+(z-a_1)1(z-a_3)\dots(z-a_n)+\dots+(z-a_1)(z-a_2)\dots(z-a_{n-1})1] =$$

$$= c[\sum_{i=1}^{n}(z-a_i)'\prod_{j\neq i}(z-a_j)] = c[\sum_{i=1}^{n}1\prod_{j\neq i}(z-a_j)] = c[\sum_{i=1}^{n}\prod_{j\neq i}(z-a_j)]$$

Then, using the above expressions, we have:

$$\int_\gamma \frac{c[\sum_{i=1}^n \prod_{j\neq i}(z-a_j)]}{c\prod_{i=1}^n (z-a_i)} = \int_\gamma \frac{\sum_{i=1}^n \prod_{j\neq i}(z-a_j)}{\prod_{i=1}^n (z-a_i)} =$$

$$= \int_\gamma \frac{(z-a_2)(z-a_3)\dots(z-a_n)}{\prod_{i=1}^n (z-a_i)} + \dots + \int_\gamma \frac{(z-a_1)(z-a_2)\dots(z-a_{n-1})}{\prod_{i=1}^n (z-a_i)} =$$

$$= \int_\gamma \frac{1}{(z-a_1)} + \int_\gamma \frac{1}{(z-a_2)} + \dots + \int_\gamma \frac{1}{(z-a_n)} = \sum_{i=1}^{n} 2\pi i\; n(\gamma, a_i) = 2\pi i n$$

$\square$