$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin(a\pi)}\ , \quad 0 < a < 1$$

Proof. To begin, consider the following substitutions: $u = e^x$ and $t^2 = u$, then:

$$\int_{\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \int_0^{\infty} \frac{u^{a-1}}{1+u} du = 2\int_0^{\infty} \frac{t^{2a-1}}{1+t^2} dt$$

We can complexify this latter integrand, and set $f(z) = \frac{z^{2a-1}}{1+z^2}$. Consider the path $\gamma = [-R,-r] \cup \gamma_r \cup [r,R] \cup \gamma_R$, where $\gamma_r(t) = re^{it}$ for $0 \le t \le \pi$ and $\gamma_R(t) = Re^{it}$ for $0 \le t \le \pi$. Then, the integral around γ can be broken up as:

$$\int_\gamma \frac{z^{2a-1}}{1+z^2} dz = \int_{-R}^{-r} \frac{x^{2a-1}}{1+x^2} dx + \int_{\gamma_r} \frac{z^{2a-1}}{1+z^2} dz + \int_r^R \frac{x^{2a-1}}{1+x^2} dx + \int_{\gamma_R} \frac{z^{2a-1}}{1+z^2} dz$$

Next, define $\log_1(z) = \log|z| + i\arg(z)$ for all z with $-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$. Then:

$$\int_{-R}^{-r} \frac{x^{2a-1}}{1+x^2} dx = \int_r^R \frac{(-x)^{2a-1}}{1+x^2} dx = \int_r^R \frac{e^{(2a-1)(\log_1(x))}}{1+x^2} dx = \int_r^R \frac{e^{(2a-1)(\log|-x|+\pi i)}}{1+x^2} dx =$$

$$= \int_r^R \frac{e^{(2a-1)(\log(x))+2a\pi i-\pi i}}{1+x^2} dx = e^{(2a-1)\pi i} \int_r^R \frac{e^{(2a-1)(\log(x))}}{1+x^2} dx = -e^{2a\pi i} \int_r^R \frac{x^{2a-1}}{1+x^2} dx$$

Next, consider the integrals:

$$\int_{\gamma_R} \frac{z^{2a-1}}{1+z^2} dz\ ,\ \textit{and} \quad \int_{\gamma_r} \frac{z^{2a-1}}{1+z^2} dz$$

When $0 < a < 1$, as per our assumption, we see that

$$\Big|\int_{\gamma_R} \frac{z^{2a-1}}{1+z^2} dz\Big| \le \int_{\gamma_R} \frac{|z|^{2a-1}}{|1+z^2|} dz \le \pi R \frac{R^{2a-1}}{-1+R^2} = \frac{\pi R^{2a}}{R^2-1} \to 0 \ \ \textit{as}\ R \to \infty$$

For the second integral, when $0 < a < 1$, as per our assumption, we have

$$\Big|\int_{\gamma_r} \frac{z^{2a-1}}{1+z^2} dz\Big| \le \int_{\gamma_r} \frac{|z|^{2a-1}}{|1+z^2|} dz \le \pi r \frac{r^{2a-1}}{-1+r^2} = \frac{\pi r^{2a}}{r^2-1} \to 0 \ \ \textit{as}\ r \to 0$$

Then, as $R \to \infty$ and $r \to 0$, we see that:

$$\int_\gamma \frac{z^{2a-1}}{1+z^2} dz = \int_0^{\infty} \frac{x^{2a-1}}{1+x^2} dx - e^{2a\pi i} \int_0^{\infty} \frac{x^{2a-1}}{1+x^2} dx$$

Then, by The Residue Theorem we know that

$$2\pi i\ Res(f,i) = \int_\gamma \frac{z^{2a-1}}{1+z^2} dz = \int_0^{\infty} \frac{x^{2a-1}}{1+x^2} dx - e^{2a\pi i} \int_0^{\infty} \frac{x^{2a-1}}{1+x^2} dx$$

because $f(z)$ has a single simple pole at $z = i$ which is enclosed by the path γ. Calculating $Res(f, i)$, we have:

$$2\pi i\ Res(f,i) = 2\pi i \lim_{z\to i} \frac{(z-i)z^{2a-1}}{(z-i)(z+i)} = \pi i^{2a-1} = \pi e^{(2a-1)\log_1(i)} = e^{\frac{-i\pi}{2}} \pi e^{a\pi i} = -\pi i e^{a\pi i}$$

and, we see that

$$-\pi i e^{a\pi i} = \int_\gamma \frac{z^{2a-1}}{1+z^2}dz = \int_0^\infty \frac{x^{2a-1}}{1+x^2}dx - e^{2a\pi i}\int_0^\infty \frac{x^{2a-1}}{1+x^2}dx \implies$$

$$\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}dx = 2(1-e^{2a\pi i})\int_0^\infty \frac{x^{2a-1}}{1+x^2}dx = -2\pi i e^{a\pi i} \implies$$

$$\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}dx = 2\int_0^\infty \frac{x^{2a-1}}{1+x^2}dx = \frac{-2\pi i e^{a\pi i}}{1-e^{2a\pi i}} = \frac{2\pi i}{e^{a\pi i} - e^{-a\pi i}} = \frac{2\pi i}{2i\sin(2\pi)} = \frac{\pi i}{\sin(2\pi)}$$

$\square$