$$\int_0^\infty \frac{x^2}{x^4+x^2+1}dx = \frac{\pi\sqrt{3}}{6}$$

Proof. Notice first that because f is an even function, we have

$$\int_0^\infty \frac{x^2}{x^4+x^2+1}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{x^4+x^2+1}dx$$

Then, consider the interval $[-R, R]$ and the half circle $\gamma_R(t) = Re^{it}$ for $0 \le t \le \pi$. Clearly, when R is sufficiently large, the complex valued function $f(z) = \frac{z^2}{z^4+z^2+1}$ has two simple poles located at $z_1 = e^{\frac{\pi i}{3}}$ and $z_2 = e^{\frac{2\pi i}{3}}$. Then, by the residue theorem we know that

$$\int_{[-R,R]\cup\gamma_R} \frac{z^2}{z^4+z^2+1}dz = \int_{-R}^{R} \frac{x^2}{x^4+x^2+1}dx + \int_\gamma \frac{z^2}{z^4+z^2+1}dz = 2\pi i\sum_{k=1}^{2} Res(f(z), z_k)$$

and therefore it follows that we have

$$\int_0^\infty \frac{x^2}{x^4+x^2+1}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{x^4+x^2+1}dx = \lim_{R\to\infty}\frac{1}{2}\int_{-R}^{R} \frac{x^2}{x^4+x^2+1}dx$$

$$= \frac{1}{2}\big[2\pi i\sum_{k=1}^{2} Res(f(z); z_k) + \lim_{R\to\infty}\int_\gamma \frac{z^2}{z^4+z^2+1}dz\big] = \frac{1}{2}\big[2\pi i\sum_{k=1}^{2} Res(f(z); z_k)\big]$$

by Jordan's Lemma since $k = 0$ and $deg(z^2) \le deg(z^4+z^2+1)+2$. Then, to calculate the values of the residues:

$$Res(f; e^{\frac{\pi i}{3}}) = \lim_{z\to e^{\frac{\pi i}{3}}}(z - e^{\frac{\pi i}{3}})\frac{z^2}{z^4+z^2+1} = \lim_{z\to e^{\frac{\pi i}{3}}}\frac{z^3 - z^2e^{\frac{\pi i}{3}}}{z^4+z^2+1} = \lim_{z\to e^{\frac{\pi i}{3}}}\frac{3z^2 - 2ze^{\frac{\pi i}{3}}}{4e^{\frac{3\pi i}{3}}+2e^{\frac{\pi i}{3}}} = \frac{e^{\frac{\pi i}{3}}}{4e^{\frac{2\pi i}{3}}+2}$$

Through a nearly identical calculation, we get

$$Res(f; e^{\frac{2\pi i}{3}}) = \frac{e^{\frac{2\pi i}{3}}}{4e^{\frac{4\pi i}{3}}+2}$$

Combining these, we get

$$\int_0^\infty \frac{x^2}{x^4+x^2+1}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{x^4+x^2+1}dx = \frac{1}{2}[2\pi i(\frac{e^{\frac{\pi i}{3}}}{4e^{\frac{2\pi i}{3}}+2} + \frac{e^{\frac{2\pi i}{3}}}{4e^{\frac{4\pi i}{3}}+2})]$$

$$= \pi i\big[\frac{e^{\frac{\pi i}{3}}}{4e^{\frac{2\pi i}{3}}+2} + \frac{e^{\frac{2\pi i}{3}}}{4e^{\frac{4\pi i}{3}}+2}\big] = \pi i\big[\frac{\frac{1}{2}+\frac{\sqrt{3}i}{2}}{-\frac{4}{2}+\frac{4\sqrt{3}i}{2}+\frac{4}{2}} + \frac{-\frac{1}{2}+\frac{\sqrt{3}i}{2}}{-\frac{4}{2}-\frac{-4\sqrt{3}i}{2}+\frac{4}{2}}\big]$$

$$= \pi i[\frac{1+\sqrt{3}i}{4\sqrt{3}i} + \frac{-1+\sqrt{3}i}{-4\sqrt{3}i}] = \frac{\pi i}{2\sqrt{3}i} = \frac{\pi\sqrt{3}}{6}$$

□