Theorem : Let f be an entire function and suppose there exists real numbers M and R, with $R > 0$, and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Then, f is a polynomial of degree less than or equal to n.

Proof. To begin, note that if f is entire, then it is analytic in any open ball $B(0, r)$ for $r > R$. Letting $r > R$, consider the following inequality, which uses the general form of Cauchy's integral formula:

$$|f^{(k)}(0)| = \Big|\frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{z^{k+1}}dz\Big| \leq \frac{k!}{2\pi}\int_\gamma \frac{|f(z)|}{|z|^{k+1}}dz \leq \frac{k!}{2\pi}\int_\gamma \frac{M|z|^n}{|z|^{k+1}}dz = \frac{k!M}{2\pi}\int_\gamma \frac{|z|^n}{|z|^{k+1}}dz =$$

$$= \frac{k!M}{2\pi}\int_\gamma |z|^{n-(k+1)}dz = \frac{k!M}{2\pi}\int_\gamma |re^{it}|^{n-(k+1)}dz = \frac{k!M}{2\pi}\int_\gamma |r|^{n-(k+1)}dz =$$

$$= \frac{k!M|r|^{n-(k+1)}}{2\pi}\int_\gamma dz = \frac{k!M|r|^{n-(k+1)}}{2\pi}(2\pi r) = Mk!r^{n-k}$$

Then, if $k > n$, we have $|f^{(k)}(0)| \leq (Mk!)/r^{-(k-n)}$ Then, because $r > 0$ was arbitrarily chosen, as $r \to \infty$, we see that:

$$|f^{(k)}(0)| \leq \frac{Mk!}{r^{k-n}} \to 0$$

Thus, whenever, $k > n$, $|f^{(k)}(0)| \to 0$. Then, to complete the proof, note that since f is entire, it can be written as a power series of the form:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

and by the above inequality, the coefficients of this power series are equal to zero whenever $k > n$ and we see that f is a polynomial of degree at most n. □