Let $A \in M_n(\mathbb{C})$, then the following are equivalent:
(i) A is normal.
(ii) $A^* = AT$ for some unitary T.
(iii) $A^* = TA$ for some unitary T.
(iv) A commutes with $AA^* - A^*A$.

Proof.

$$(i \to ii)$$

Let A be normal, then there exists a unitary matrix, U, such that

$$A = U^* D_A U = U^* \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} U$$

Then, define:

$$T = U^* D_u U = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix}$$

where $\mu_i = \frac{\overline{\lambda_i}}{\lambda_i}$ whenever $\lambda_i \neq 0$ and $\mu_i = 1$ whenever $\lambda_i = 0$. Clearly, T is unitary. Then:

$$AT = U^* \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} UU^* \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix} U = U^* \begin{pmatrix} \lambda_1\mu_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n\mu_n \end{pmatrix} U =$$

$$U^* \begin{pmatrix} \overline{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \overline{\lambda_n} \end{pmatrix} U = U^* D_A^* U = (U^* D_A U)^* = A^*$$

$$(ii \to i)$$

Assume that there exists a unitary matrix, T, such that $A^* = AT$, then: $A^*A = A^*(A^*)^* = AT(AT)^* = ATT^*A^* = AA^*$ and A is therefore normal.

$$(i \to iv)$$

Assume that A is normal, then:

$$A(AA^* - A^*A) = AAA^* - AA^*A = A(A^*A) - (A^*A)A = AA^*A - A^*AA = (AA^* - A^*A)A$$

$$(iv \to i)$$

Assume that $A(AA^* - A^*A) = (AA^* - A^*A)A$, then:

$$AAA^* - AA^*A = AA^*A - A^*AA \implies$$
$$(AAA^*)A^* - (AA^*A)A^* = (AA^*A)A^* - (A^*AA)A^* \implies$$
$$A^2(A^*)^2 - (AA^*)^2 = (AA^*)^2 - A^*AAA^*$$

Then, taking the trace of both sides, we have:

$$tr(A^2(A^*)^2) - tr((AA^*)^2) = tr((AA^*)^2) - tr(A^*AAA^*) \implies$$

$$tr(A^2(A^*)^2)+tr(A^*AAA^*) = 2tr((AA^*)^2) \implies tr(A^2(A^*)^2)+tr(AAA^*A^*) = 2tr((AA^*)^2) \implies$$

$$2tr(A^2(A^*)^2) = 2tr((AA^*)^2) \implies tr(A^2(A^*)^2) = tr((AA^*)^2)$$

which, by theorem (in-class) implies that $A^*A = AA^*$ and therefore A is normal.

$$(iii \to i)$$

Assume that there exists a unitary matrix T such that $A^* = TA$, then: $AA^* = (A^*)^*A^* = (TA)^*(TA) = A^*T^*TA = A^*A$ and A is therefore normal.

$$(i \to iii)$$

Let A be normal, then there exists a unitary matrix, U, such that

$$A = U^*D_AU = U^* \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} U$$

Then, define:

$$T = U^*D_uU = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix}$$

where $\mu_i = \frac{\overline{\lambda_i}}{\lambda_i}$ whenever $\lambda_i \neq 0$ and $\mu_i = 1$ whenever $\lambda_i = 0$. Clearly, T is unitary. Then:

$$TA = U^* \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix} UU^* \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} U = U^* \begin{pmatrix} \mu_1\lambda_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n\lambda_n \end{pmatrix} U =$$

$$U^* \begin{pmatrix} \overline{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \overline{\lambda_n} \end{pmatrix} U = U^*D_A^*U = (U^*D_AU)^* = A^*$$

$\square$